Topological Interference Alignment in Wireless Networks

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joint work with Kishore Jaganathan and Christos Thramboulidis

California Institute of Technology

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Outline

- Interference Alignment
  - degrees-of-freedom
  - channel state issues, ergodic interference alignment

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  - finite SNR
  - efficient algortihms

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- Conclusion

Wireless Networks

As we all know, wireless communication systems are characterized by

1. broadcast during transmission
2. interference during reception
3. random fading
4. path-loss
5. mobility and time-varying channel conditions
6. time-varying traffic patterns

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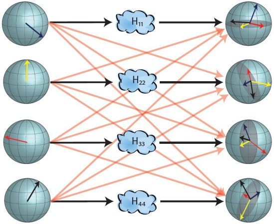
All have been successfully expolited in practical systems (perhaps) with the exception of *interference*.

Interference Channels



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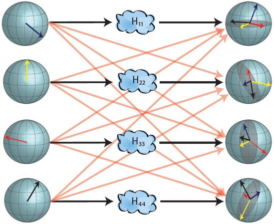

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Focus, instead, on degrees-of-freedom:

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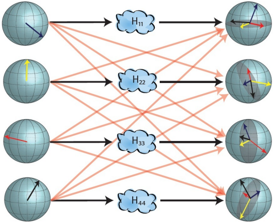

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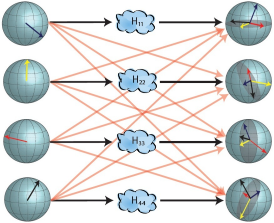

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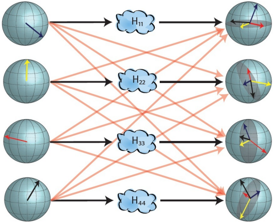

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  - considerably simplifies the analysis
  - can lead to physical insight
- Cons:
  - may not "well reflect" actual performance at practical SNRs

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- Assume the channel coefficients change over time: $y_i(t) = h_{ii}(t)x_i(t) + \sum_{j\neq i} h_{ij}(t)x_j(t) + z_j(t)$
- Consider T channel uses:

$$\underbrace{\begin{bmatrix} y_i(1) \\ \vdots \\ y_i(T) \end{bmatrix}}_{Y_i} = \underbrace{\begin{bmatrix} h_{ii}(1) & & \\ & \ddots & \\ & & h_{ii}(T) \end{bmatrix}}_{H_{ii}} \underbrace{\begin{bmatrix} x_i(1) \\ \vdots \\ x_i(T) \end{bmatrix}}_{X_i} +$$

$$\sum_{j\neq i} \underbrace{\begin{bmatrix} h_{ij}(1) & & \\ & \ddots & \\ & & h_{ij}(T) \end{bmatrix}}_{H_{ij}} \underbrace{\begin{bmatrix} x_j(1) \\ \vdots \\ x_j(T) \end{bmatrix}}_{X_j} + \underbrace{\begin{bmatrix} z_i(1) \\ \vdots \\ z_i(T) \end{bmatrix}}_{Z_i}$$

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$$X_j = V_j S_j,$$

where $V_j \in \mathcal{C}^{T \times m}$ represents the precoding matrix.

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$$\begin{bmatrix} H_{i1}V_1 & \dots & H_{i,i-1}V_{i-1} & H_{i,i+1}V_{i+1} & \dots & H_{in}V_n \end{bmatrix}_{T \times (n-1)m}.$$

Interference Alignment (Cadambe and Jafar, 2008)

If we can find precoding matrices $V_i \in \mathcal{C}^{T\times m}$ and decoding matrices $U_i \in \mathcal{C}^{m\times T}$ such that

1. $\mathrm{rank}(U_i H_{ii} V_i) = m$
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When $T = n$, $m = 1$ is trivially achieved by time sharing. ($DoF = 1$.)

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As an optimization problem

$$\max \quad \text{rank} \left(\underbrace{\begin{bmatrix} U_1 & & \\ & \ddots & \\ & & U_n \end{bmatrix} \begin{bmatrix} H_{11} & \dots & H_{1n} \\ \vdots & \ddots & \vdots \\ H_{n1} & \dots & H_{nn} \end{bmatrix} \begin{bmatrix} V_1 & & \\ & \ddots & \\ & & V_n \end{bmatrix}}_{A} \right)$$

$$\text{subject to} \quad A = \begin{bmatrix} \times & & \\ & \ddots & \\ & & \times \end{bmatrix}$$

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Cadambe and Jafar's argument relies heavily on the fact that the H_{ij} are diagonal. They give explicit constructions for the precoding matrices when $T = O(n^N)$.

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This is clearly not practically feasible. (But it does suggest what to shoot for in practical systems.)

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Nonetheless, there is a growing literature on attempting to do interference alignment with more reasonable CSIT assumptions. (The jury is still out on what the gains are.)

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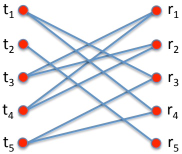
Example:

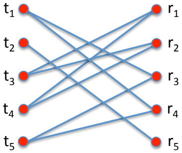


(a) Interference pattern

$$\begin{bmatrix} 1 & \times & 0 & 0 & \times \\ \times & 1 & 0 & 0 & \times \\ 0 & \times & 1 & \times & 0 \\ 0 & \times & \times & 1 & 0 \\ \times & 0 & \times & \times & 1 \end{bmatrix}$$

(b) Matrix entry pattern

Interference Avoidance (Graph Coloring)



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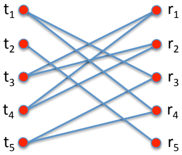

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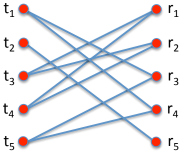

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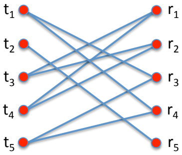

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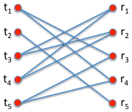
$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Topological Interference Alignment



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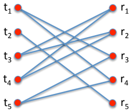

Let each transmitter transmit one signal over two channel uses each:

$$X_1 = \begin{bmatrix} s_1 \\ 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 \\ s_2 \end{bmatrix},\ X_3 = \begin{bmatrix} -s_3 \\ s_3 \end{bmatrix},\ X_4 = \begin{bmatrix} -s_4 \\ s_4 \end{bmatrix},\ X_5 = \begin{bmatrix} s_5 \\ 0 \end{bmatrix}.$$

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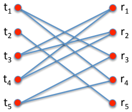

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Y_1, Y_3 and Y_5 therefore are

$$\begin{aligned} Y_1 &= \begin{bmatrix} s_1 \\ 0 \end{bmatrix} h_{11} + \begin{bmatrix} -s_3 \\ s_3 \end{bmatrix} h_{13} + \begin{bmatrix} -s_4 \\ s_4 \end{bmatrix} h_{14} + Z_1 \\ Y_3 &= \begin{bmatrix} -s_3 \\ s_3 \end{bmatrix} h_{33} + \begin{bmatrix} s_1 \\ 0 \end{bmatrix} h_{31} + \begin{bmatrix} s_5 \\ 0 \end{bmatrix} h_{35} + Z_3 \\ Y_5 &= \begin{bmatrix} s_5 \\ 0 \end{bmatrix} h_{55} + \begin{bmatrix} 0 \\ s_2 \end{bmatrix} h_{52} + Z_5 \end{aligned}$$

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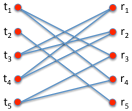

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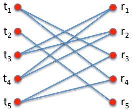
Topological Interference Alignment



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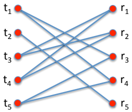

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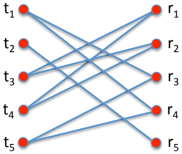

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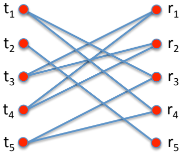

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Key Concept

S: set of all pairs (i, j) such that receiver i has interference from transmitter j

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Literature:

- Lots of attention in compressed-sensing and machine learning communities [Fazel, Recht, Parrilo, Candes, Montanari, Sanghavi, Oymak-Hassibi, etc.]

Nuclear Norm Minimization

- The non-convex optimization problem

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Instead of searching for the optimal r, seek a completion for a *fixed* r:

Matrix Completion Problem:

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Observation: It is very easy to project any given matrix onto the sets (S1) and (S2) individually

Alternating Projection Method

Algorithm 1 Proposed Algorithm: Alternating Projection Method

Let A^0 be a random matrix. From $i = 0$ until convergence:

- Project A^i onto (S1): $B^i = svd(A^i, r)$
- Project B^i onto (S2): $A^{i+1} = [B^i]_{S^c} + I$

Descent method:

- B^{i+1} is the best rank r approximation of $[B^i]_{S^c} + I$

$$||B^{i+1} - ([B^i]_{S^c} + I)||_F^2 \leq ||B^i - ([B^i]_{S^c} + I)||_F^2$$

$$\Rightarrow ||B_{S^c}^{i+1} - B_{S^c}^i||_F^2 + ||B_S^{i+1} - I||_F^2 \leq ||B_S^i - I||_F^2$$

Convergence to fixed points:

$$B = svd(B_{S^c} + I, r)$$

Alternating Minimization

Algorithm 2 AltMin

Inputs: n, r, S, P_t. **Initialization**: $U_0 \in \mathcal{R}^{n \times r}$ random.
From $i = 0$ until convergence,

- Solve for V_i:

$$\text{minimize} \qquad \|(U_{i-1} V_i^T - I)_S\|$$

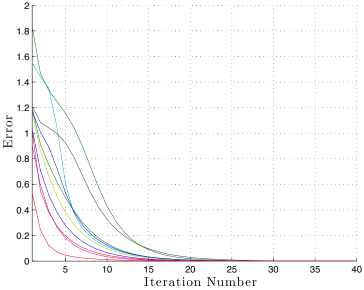
- Solve for U_i:

$$\text{minimize} \qquad \|(U_i V_i^T - I)_S\|$$

If algorithm converges to V_N and U_N,
output V_N and U_N.

S includes the set of indices where $A_{ij} = 0$ and the diagonal.

Numerical Experiments



$$M = \begin{bmatrix} 1 & -2.09 & 0 & 0 & 0.81 \\ -0.47 & 1 & 0 & 0 & -0.39 \\ 0 & 1.73 & 1 & 0.69 & 0 \\ 0 & 2.52 & 1.45 & 1 & 0 \\ 1.23 & 0 & 1.49 & 1.03 & 1 \end{bmatrix} = \begin{bmatrix} 0.93 & 0.89 \\ -0.44 & -0.42 \\ -1.00 & 0.17 \\ -1.46 & 0.25 \\ -0.35 & 1.35 \end{bmatrix} \begin{bmatrix} 0.26 & 0.96 \\ -1.80 & -0.47 \\ -0.84 & 0.89 \\ -0.58 & 0.61 \\ 0.13 & 0.77 \end{bmatrix}^T$$

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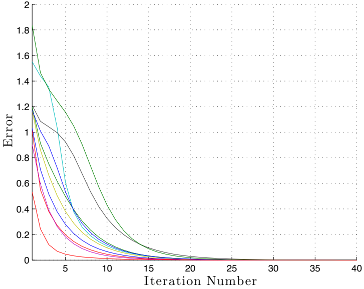

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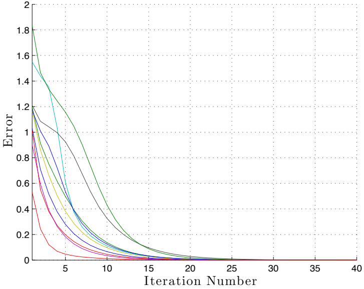

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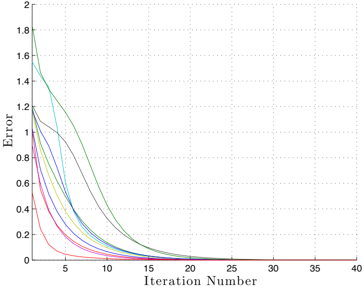

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Digression: Low Rank Matrix Completion over Finite Fields

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The problems of

1. network coding
2. index coding
3. distributed storage
4. secret sharing

for **linear codes**, can all be recast as the problem

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where the elements of A and Y belong to some *finite field* $\mathcal{F}_q$.

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- While there is a huge literature on matrix completion over the real and complex fields, there is virtually no literature for finite fields.
- Can one leverage the former results for the latter? (Compressed sensing and LP decoding.)

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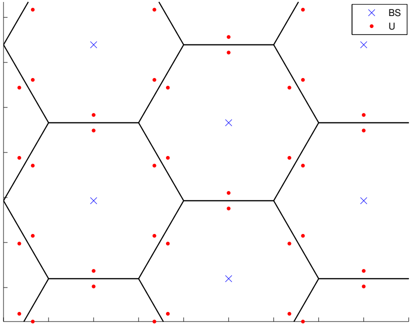
- How is the capacity affected when you consider geometrically-placed transmitters and receivers, path-loss models, fading and put back in the real channel coefficients?
- How does TIM compare to the baseline, i.e., *interference avoidance* (frequency reuse, etc)?

Hexagonal Grid: Setup

- N=8,18,24,32,50 cells.
- 6 users per cell,
- average SNR in each cell $= 20db$
- average INR from neighboring cell $= 12db$
- path loss model: $h_{ij} \sim \mathcal{N}(0, \left(\frac{d_{ij}}{r_0}\right)^{-4.0})$



Methods

1. frequency reuse 3 yields $DoF = \frac{1}{18}$
2. with carefully-placed users, and no fading, Jafar exhibits an optimal $DoF = \frac{1}{7}$ (257% improvement)
3. we will randomly place 6 users in each cell and will consider fading

Hexagonal Grid: Results

- DoF

	FreqReuse	Coloring	AltMin
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$N = 18$	23.3473	23.1307	13.0369
$N = 24$	29.0311	29.2044	14.9266
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This is really bad. What is going on?

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- At receiver i

$$u_i y_i = u_i v_i h_{ii} s_i + \sum_{j:A_{ij}=0}^{n} \underbrace{u_i v_j}_{=0} h_{ij} s_j + \sum_{j:A_{ij}=\times}^{n} u_i v_j h_{ij} s_j + u_i z_i.$$

Let us Look at the Sum Rate

- Transmitter i has signal s_i, $E|s_i|^2 = 1$ and transmits $x_i = v_i s_i \in \mathcal{R}^r$. The power constraint per channel use is $\frac{E\|x_i\|^2}{r} = P_t$, which translates to $\|v_i\|^2 = rP_t$.
- At receiver i

$$u_i y_i = u_i v_i h_{ii} s_i + \sum_{j:A_{ij}=0}^{n} \underbrace{u_i v_j}_{=0} h_{ij} s_j + \sum_{j:A_{ij}=\times}^{n} u_i v_j h_{ij} s_j + u_i z_i.$$

- Therefore the sum rate is

$$C_{sum} = \sum_{i=1}^{r} \frac{1}{r} \log \left(1 + \frac{|u_i v_i|^2 |h_{ii}|^2}{\sigma^2 \|u_i\|^2 + \sum_{j:A_{ij}=\times}^{n} |u_i v_j|^2 |h_{ij}|^2} \right)$$

or

$$C_{sum} = \sum_{i=1}^{r} \frac{1}{r} \log \left(1 + \frac{\frac{|u_i v_i|^2}{\|u_i\|^2 \|v_i\|^2} rP_t |h_{ii}|^2}{\sigma^2 + \sum_{j:A_{ij}=\times}^{n} \frac{|u_i v_j|^2}{\|u_i\|^2 \|v_j\|^2} rP_t |h_{ij}|^2} \right)$$

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- Looking at the results of the simulations for "AltMin", the value $\frac{|u_i v_i|^2}{\|u_i\|^2 \|v_i\|^2}$ was often very small.
- Therefore we will impose the extra constraint in the algorithm that

$$\frac{|u_i v_i|^2}{\|u_i\|^2 \|v_i\|^2} \geq c, \qquad \text{for some } 0 \leq c \leq 1.$$

Constrained Alternating Minimization

Algorithm 3 AltMinCon

Inputs: n, r, S, c, P_t. **Initialization**: $U_0 \in \mathcal{R}^{n\times r}$ random.
From $i = 0$ until convergence,

- Solve for V_i:

$$\begin{array}{ll} \text{minimize} & \|(U_{i-1}V_i)_S\| \\ \text{subject to} & \|\mathbf{v}_j^{(i)}\| \leq 1 \text{ and } (\mathbf{u}_j^{(i-1)})^T\mathbf{v}_j^{(i)} \geq c\|\mathbf{u}_j^{(i-1)}\| \quad \forall j \end{array}$$

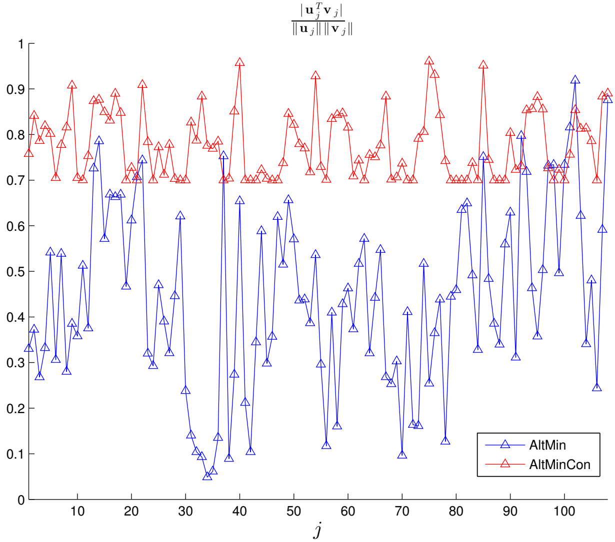
- Solve for U_i:

$$\begin{array}{ll} \text{minimize} & \|(U_iV_i)_S\| \\ \text{subject to} & \|\mathbf{u}_j^{(i)}\| \leq 1 \text{ and } (\mathbf{u}_j^{(i)})^T\mathbf{v}_j^{(i)} \geq c\|\mathbf{v}_j^{(i)}\| \quad \forall j \end{array}$$

If algorithm converges to V_N and U_N,
normalize columns of V_N to satisfy transmit power constraint $\|\mathbf{v}_j^{(N)}\| \leq \sqrt{r}P_t$.
output V_N and U_N.

S includes only the set of indices where $A_{ij} = 0$.

AltMin vs AltMinCon



Hexagonal Grid: Results

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- DoF

	FreqReuse	Coloring	AltMin	AltMinCon
DoF	$1/18$	$1/11$	$1/9$	$1/11$

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	FreqReuse	Coloring	AltMin	AltMinCon
$N = 8$	13.5302	14.7916	6.2415	11.3251
$N = 18$	23.3473	23.1307	13.0369	20.6579
$N = 24$	29.0311	29.2044	14.9266	23.7311
$N = 32$	41.2803	39.0702	22.3766	34.6017
$N = 50$	60.4578	62.7105	35.1663	54.3691

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Better, but still not quite good enough. What is going on?

Back to the Sum Rate

$$C_{sum} = \sum_{i=1}^{r} \frac{1}{r} \log \left(1 + \frac{\frac{|u_i v_i|^2}{\|u_i\|^2 \|v_i\|^2} r P_t |h_{ii}|^2}{\sigma^2 + \sum_{j:A_{ij}=\times}^{n} \frac{|u_i v_j|^2}{\|u_i\|^2 \|v_j\|^2} r P_t |h_{ij}|^2} \right)$$

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We therefore propose

$$\min_{U \in \mathcal{R}^{n\times r}, V \in \mathcal{R}^{r\times n}} \sum_{(i,j)\in S, i\neq j} |u_i v_j|^2 + \lambda \sum_{(i,j)\notin S} |u_i v_j|^2 E|h_{ij}|^2$$

where $E|h_{ij}|^2$ depends only on the (distance of the) cells in which receiver i and transmitter j live, subject to

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- The above can also be solved in an alternating minimization fashion.

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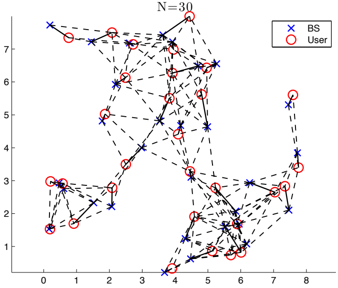
We get 10%-20% improvement in the sum rate

Ad hoc Network Example

- N=100 Tx-Rx pairs randomly placed in a 20×20 square
- max distance btw Tx-Rx is 1
- average SNR to desired user $= 20db$
- path loss model: $h_{ij} \sim \mathcal{N}(0, \left(\frac{d_{ij}}{r_0}\right)^{-4.0})$



Algorithms

1. greedy Coloring (Coloring)
2. matrix Completion (AltMin)
3. constrained matrix Completion (AltMinCon)
4. rate optimization (RateOpt)

Ad hoc Network Results

- Average values over 25 realizations

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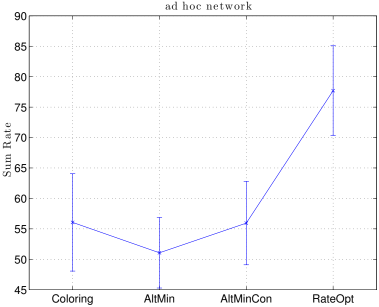

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Rank	6.28	6.16	6.16	3.28
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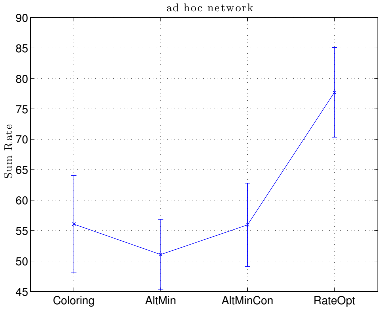

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We obtain a %40 improvement in the sum rate.

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  - requires only topological information of the network; can significantly improve the DoF
  - reduces to low rank matrix completion
  - related to network coding, index coding, secret sharing (when over finite fields)
- In practice DoF can be misleading
  - developed alternative algorithms (moved away somewhat from TIM)
  - promising preliminary results: there is something to be had

Possible Future Work

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- Algorithmic issues: theoretical analysis, fast implementation
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